Two-parameter Quantum Groups of Exceptional Type E-Series and Convex PBW-Type Basis *

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Abstract

The presentation of two-parameter quantum groups of type E-series in the sense of Benkart-Witherspoon [BW1] is given, which has a Drinfel'd quantum double structure. The universal R-matrix and a convex PBW-type basis are described for type E_6 (as a sample), and the conditions of those isomorphisms from these quantum groups into the one-parameter quantum doubles are discussed.

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Introduction

Two-parameter or multiparameter quantum groups have been investigated by many authors (see the references in [BW1], [BGH1], etc.). From another viewpoint based on the work on down-up algebras (see [B]), Benkart and Witherspoon [BW1] recovered the structure of two-parameter quantum enveloping algebras of the general linear Lie algebra $\mathfrak{gl}_n$ and the special linear Lie algebra $\mathfrak{sl}_n$, which was earlier gotten by Takeuchi [T]. They studied their finite-dimensional weight representation theory in the case when rs^{-1} is not a root of unity ([BW2]) and the restricted quantum version (or say, the small quantum groups in the two-parameter setting) at rs^{-1} being a root of unity ([BW3]). Inspired by their work, the two-parameter quantum groups in the sense of Benkart-Witherspoon corresponding to the orthogonal Lie algebras $\mathfrak{so}_{2n+1}$ or $\mathfrak{so}_{2n}$ and the symplectic Lie algebras $\mathfrak{sp}_{2n}$, as well as the exceptional type G_2 were further obtained by Bergeron-Gao-Hu [BGH1] and Hu-Shi [HS], respectively. Their finite-dimensional weight representation theory and Lusztig symmetries' property were systematically established in [BGH2] and [HS]. Actually, this kind of Lusztig symmetries' property existing from these quantum groups to their associated objects also

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reveals the difference with the standard Drinfeld-Jimbo quantum groups in the one-parameter setting (see [Ja]).

The aim of this paper is to give the presentation of two-parameter quantum groups of exceptional type E-series, to describe the universal R-matrix and a convex PBW-type basis in terms of Lyndon words (cf. [LO]), as well as to study those isomorphisms' conditions from these quantum groups into the one-parameter quantum doubles. Here we will give a general formalism (see Section 1) of the presentation of their structural constants, which is actually applied to all simply-laced types (including types A, D).

Let $\mathfrak{g}$ denote one of Lie algebras of type E_6, E_7, or E_8, and $U_{r,s}(\mathfrak{g})$, the two-parameter quantum enveloping algebra of $\mathfrak{g}$. For simplicity, we will only write down the results of E_6 in this paper (and those for E_7 and E_8 are similar to be obtained).

1. Presentation of two-parameter quantum group of type E

Consider the root system of E_6 as a root subsystem of E_8. Assume Φ is a finite root system of type E_6 with a base of simple roots Π. We regard Φ as a subset of a Euclidean space $\mathbb{R}^8$ with an inner product $(\ ,\ )$. Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_8$ denote an orthonormal basis of $\mathbb{R}^8$, and suppose $\Pi = \{\alpha_1 = \frac{1}{2}(\epsilon_1+\epsilon_8) - \frac{1}{2}(\epsilon_2+\cdots+\epsilon_7),\ \alpha_2 = \epsilon_1+\epsilon_2,\ \alpha_j = \epsilon_{j-1}-\epsilon_{j-2} \mid 3 \le j \le 6\}$ and $\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \le j \ne i \le 5\} \cup \{\pm\frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5} \pm\epsilon_i) \mid$ even number of minus signs $\}$.

Fix two nonzero elements r, s in a field $\mathbb{K}$ with $r \ne s$.

Let $U = U_{r,s}(\mathfrak{g})$ be the unital associative algebra over $\mathbb{K}$ generated by elements $e_j, f_j, \omega_i^{\pm 1}, \omega_i'^{\pm 1} (1 \le i \le 6)$, which satisfy the following relations:

(E1) $[\omega_i^{\pm 1}, \omega_j^{\pm 1}] = 0 = [\omega_i'^{\pm 1}, \omega_j'^{\pm 1}] = [\omega_i^{\pm 1}, \omega_j'^{\pm 1}]$, $\omega_i\omega_i^{-1} = \omega_j'\omega_j'^{-1} = 1$.

(E2) For $1 \le i, j \le 6$, we have

$$\omega_i e_j \omega_i^{-1} = r^{p_{ij}}(s^{-1})^{q_{ij}} e_j, \qquad \omega_i f_j \omega_i^{-1} = (r^{-1})^{p_{ij}} s^{q_{ij}} f_j,$$

where $p_{ij} + q_{ij} = (\alpha_i, \alpha_j)$, $p_{ij}, q_{ij} \in \{0, \pm 1\}$, and if $(\alpha_i, \alpha_j) \ne 0$, then $p_{ij} - q_{ij}$, $j - i$ have the same sign.

(E3) $\omega'_i e_j \omega_i'^{-1} = s^{p_{ij}}(r^{-1})^{q_{ij}} e_j, \qquad \omega'_i f_j \omega_i'^{-1} = (s^{-1})^{p_{ij}} r^{q_{ij}} f_j.$

(E4) For $1 \le i, j \le 6$, we have

$$[e_i, f_j] = \frac{\delta_{i,j}}{r-s}(\omega_i - \omega_i').$$

(E5) For $1 \le i, j \le 6$, and $(\alpha_i, \alpha_j) = 0$,

$$[e_i, e_j] = [f_i, f_j] = 0.$$

(E6) For $1 \le i < j \le 6$, with $a_{ij} = -1$, we have

$$e_i^2 e_j - (r+s) e_i e_j e_i + (rs) e_j e_i^2 = 0,$$
$$e_j^2 e_i - (r^{-1}+s^{-1}) e_j e_i e_j + (r^{-1}s^{-1}) e_i e_j^2 = 0.$$

(E7) For $1 \leq i < j \leq 6$, with $a_{ij} = -1$, we have

$$f_j f_i^2 - (r+s) f_i f_j f_i + (rs) f_i^2 f_j = 0,$$
$$f_i f_j^2 - (r^{-1}+s^{-1}) f_j f_i f_j + (r^{-1}s^{-1}) f_j^2 f_i = 0.$$

Remark. It is easy to see that when $(\alpha_i, \alpha_j) = 0$, we have two solutions of the equation $p_{ij} + q_{ij} = (\alpha_i, \alpha_j)$, that is, $p_{ij} = q_{ij} = 0$ and $p_{ij} = \pm 1, q_{ij} = \mp 1$. We have checked that both of them work, but later on in the next section we only discuss the case when $p_{ij} = q_{ij} = 0$ for simplicity. Then for any fixed (i, j), p_{ij} and q_{ij} can be determined uniquely.

Lemma 1.1. *For any simply-laced simple Lie algebra, there hold identities*: $p_{ij} = q_{ji}$.

Proof. Notice that $p_{ij} + q_{ij} = (\alpha_i, \alpha_j), \quad p_{ji} + q_{ji} = (\alpha_j, \alpha_i)$. Since $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)$, then $\{p_{ij}, q_{ij}\}$ and $\{p_{ji}, q_{ji}\}$ are all the solution of the same equation. Assume that $i > j$ are two fixed integers, then $\{p_{ij} \leq q_{ij}\}$ and $\{p_{ji} \geq q_{ji}\}$. Since the solution is determined uniquely, we can deduce that $\{p_{ij}, q_{ij}\} = \{p_{ji}, q_{ji}\}$ and $p_{ij} = q_{ji}, q_{ij} = p_{ji}$. So we get the result. $\square$

Let $\mathcal{B} = \mathcal{B}(\mathfrak{g})$ (resp. $\mathcal{B}' = \mathcal{B}'(\mathfrak{g})$) denote the Hopf subalgebra of $U = U_{r,s}(\mathfrak{g})$, which is generated by $e_j, \omega_j^{\pm}$ (resp. $f_j, {\omega'}_j^{\pm}$), where $1 \leq i \leq 6$. Then we have

Proposition 1.2. *The algebra $U_{r,s}(\mathfrak{g})$ is a Hopf algebra under the comultiplication, the counit and the antipode below*

$$\Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1}, \quad \Delta({\omega'}_i^{\pm 1}) = {\omega'}_i^{\pm 1} \otimes {\omega'}_i^{\pm 1},$$
$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes {\omega'}_i,$$
$$\varepsilon(\omega_i^{\pm 1}) = \varepsilon({\omega'}_i^{\pm 1}) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$
$$S(\omega_i^{\pm 1}) = \omega_i^{\mp 1}, \quad S({\omega'}_i^{\pm 1}) = {\omega'}_i^{\mp 1},$$
$$S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i {\omega'}_i^{-1}.$$

We can define the left-adjoint and the right-adjoint action in Hopf algebra $U_{r,s}(\mathfrak{g})$ as follows

$$ad_\ell\, a(b) = \sum_{(a)} a_{(1)} b\, S(a_{(2)}), \qquad ad_r a(b) = \sum_{(a)} S(a_{(1)}) b\, a_{(2)},$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$, $a,\, b \in U_{r,s}(\mathfrak{g})$.

Let $U_{r,s}(\mathfrak{n})$ (resp. $U_{r,s}(\mathfrak{n}^-)$) denote the subalgebra of $\mathcal{B}$ (resp. $\mathcal{B}'$) generated by e_i (resp. f_i) for all $1 \leq i \leq 6$. Let

$$U^0 = \mathbb{K}[\omega_1^{\pm}, \cdots, \omega_6^{\pm}, {\omega'}_1^{\pm}, \cdots, {\omega'}_6^{\pm}],$$
$$U_0 = \mathbb{K}[\omega_1^{\pm 1}, \cdots, \omega_6^{\pm}], \qquad U_0' = \mathbb{K}[{\omega'}_1^{\pm 1}, \cdots, {\omega'}_6^{\pm}],$$

denote the respective Laurent polynomial subalgebras of $U_{r,s}(\mathfrak{g})$, $\mathcal{B}$ and $\mathcal{B}'$. Then we have $\mathcal{B} = U_{r,s}(\mathfrak{n}) \rtimes U_0$, and $\mathcal{B}' = U_0' \ltimes U_{r,s}(\mathfrak{n}^-)$.

Similar to the type A case (see [BW1]), we have

Proposition 1.3. *There exists a unique skew-dual pairing $\langle\,,\rangle : \mathcal{B}'(\mathfrak{g})\times\mathcal{B}(\mathfrak{g}) \longrightarrow \mathbb{Q}(r,s)$ of the Hopf algebras $B(\mathfrak{g})$ and $B'(\mathfrak{g})$ such that*

$$\langle f_i, e_j\rangle = \delta_{ij}\frac{1}{s-r}, \tag{1.1}$$

$$\langle \omega_i', \omega_j\rangle = r^{p_{ji}}(s^{-1})^{q_{ji}}, \tag{1.2}$$

$$\langle {\omega'}_i^{\pm}, {\omega_j}^{-1}\rangle = \langle {\omega'}_i^{\pm}, \omega_j\rangle^{-1} = \langle \omega_i', \omega_j\rangle^{\mp 1}, \tag{1.3}$$

and all other pairs of generators are 0. Moreover, we have $\langle S(a), S(b)\rangle = \langle a, b\rangle$ for $a\in\mathcal{B}'$, $b\in\mathcal{B}$.

As a result of Proposition 1.3, we can display the structural constants for type E_6 by a matrix $A = (\tilde{a}_{ij})$, where $\tilde{a}_{ij} = \langle {\omega'}_i, \omega_j\rangle$,

$$A = \begin{pmatrix} rs^{-1} & 1 & r^{-1} & 1 & 1 & 1 \\ 1 & rs^{-1} & 1 & r^{-1} & 1 & 1 \\ s & 1 & rs^{-1} & r^{-1} & 1 & 1 \\ 1 & s & s & rs^{-1} & r^{-1} & 1 \\ 1 & 1 & 1 & s & rs^{-1} & r^{-1} \\ 1 & 1 & 1 & 1 & s & rs^{-1} \end{pmatrix}.$$

Proposition 1.4. ([BGH1, Coro. 2.7]) *For $\zeta = \sum_{i=1}^{6}\zeta_i\alpha_i \in Q$, the defining relations $(E2)$ in $U_{r,s}(\mathfrak{g})$ can be rewritten as the forms below*

$$\omega_\zeta e_i \omega_\zeta^{-1} = \langle {\omega'}_i, \omega_\zeta\rangle e_i, \qquad \omega_\zeta f_i \omega_\zeta^{-1} = \langle {\omega'}_i, \omega_\zeta\rangle^{-1} f_i,$$
$$ {\omega'}_\zeta e_i {\omega'}_\zeta^{-1} = \langle {\omega'}_\zeta, \omega_i\rangle^{-1} e_i, \qquad {\omega'}_\zeta f_i {\omega'}_\zeta^{-1} = \langle {\omega'}_\zeta, \omega_i\rangle f_i.$$

Then $U_{r,s}(\mathfrak{g}) = \bigoplus_{\eta\in Q} U_{r,s}^{\eta}(\mathfrak{g})$ is Q-graded such that

$$\begin{aligned} U_{r,s}^{\eta}(\mathfrak{g}) \;=\; \Big\{ \sum F_\alpha {\omega'}_\mu \omega_\nu E_\beta \in U \;\Big|\; & \omega_\zeta(F_\alpha {\omega'}_\mu \omega_\nu E_\beta)\omega_\zeta^{-1} = \langle {\omega'}_{\beta-\alpha}, \omega_\zeta\rangle F_\alpha {\omega'}_\mu \omega_\nu E_\beta, \\ & {\omega'}_\zeta(F_\alpha {\omega'}_\mu \omega_\nu E_\beta){\omega'}_\zeta^{-1} = \langle {\omega'}_\zeta, \omega_{\beta-\alpha}\rangle^{-1} F_\alpha {\omega'}_\mu \omega_\nu E_\beta, \text{ with } \beta-\alpha=\eta \Big\}, \end{aligned}$$

where F_α (resp. E_α) is a certain monomial $f_{i_1}\cdots f_{i_l}$ (resp. $e_{i_1}\cdots e_{i_m}$) such that $\alpha_{i_1} + \cdots + \alpha_{i_l} = \alpha$ (resp. $\alpha_{j_1}+\cdots+\alpha_{j_m} = \beta$).

2. Lyndon words and convex PBW-type basis

Thanks to the work in [LR], [K1,2] and [R2,3], there is a combinatorial approach to constructing an ordered basis called a convex PBW-type basis (for definition, see [R3]) for our $U_{r,s}(\mathfrak{n})$. In this section, we will give a description of a convex PBW-type basis of $U_{r,s}(\mathfrak{n})$ making use of Lyndon words and (r,s)-bracketing.

Let $A=\{e_1,e_2,\cdots,e_6\}$ be an ordered alphabet set and the order is defined by $e_1<e_2<\cdots<e_6$. Let A^* be the set of all words in the alphabet set A and let $u<v$ denote that word u is lexicographically smaller than word v.

Definition 2.1. *A word $\ell\in A^*$ is a Lyndon word if it is lexicographically smaller than all its proper right factors.*

Definition 2.2. *Let $\ell=uv$, we call it a Lyndon decomposition if u, v are both Lyndon words and u is the shortest Lyndon word appearing as a proper left factor of ℓ.*

Let $\mathbb{K}[A^*]$ be the associative algebra of $\mathbb{K}$-linear combinations of words A^* whose product is juxtaposition, namely, a free $\mathbb{K}$-algebra.

Theorem 2.3.([LR], [R2,3]) *The set of products $\ell_1\cdots\ell_k$ is a basis of $\mathbb{K}[A^*]$, where the ℓ_i's are Lyndon words and $\ell_1\geq\cdots\geq\ell_k$.*

Let J be the (r,s)-Serre ideal of $\mathbb{K}[A^*]$ generated by elements $\{(ad_\ell e_i)^{1-a_{ij}}(e_j)\mid 1\leq i\neq j\leq 6\}$. Now it is clear that $U_{r,s}(\mathfrak{n})=\mathbb{K}[A^*]/J$.

In order to construct a monomial basis of $U_{r,s}(\mathfrak{n})$, we need to give another kind of order $\preceq$ in A^* with introducing a usual length function $|\cdot|$ for a word $u\in A^*$. We say $u\preceq w$, if $|u|<|w|$ or $|u|=|w|$ and $u\geq w$.

Definition 2.4. *Call a (Lyndon) word to be good w.r.t. the (r,s)-Serre ideal J if it cannot be written as a sum of strictly smaller words modulo J w.r.t. the ordering $\preceq$.*

For example, e_1e_2 is not "good", since $e_1e_2=e_2e_1$ and e_2e_1 is strictly smaller than e_1e_2 w.r.t. to the ordering $\preceq$.

Theorem 2.5. *The set of products $\ell_1\cdots\ell_k$, where ℓ_i's are good Lyndon words and $\ell_1\geq\cdots\geq\ell_k$, is a basis of $U_{r,s}(\mathfrak{n})$ (Set $U^+:=U_{r,s}(\mathfrak{n})$ for short).*

Proof. First, we claim that the set of good words is a basis for $U_{r,s}(\mathfrak{n})=\mathbb{K}[A^*]/J$. Every element in $\mathbb{K}[A^*]/J$ can be written as a linear combination of the words in $\mathbb{K}[A^*]$ and if any of them is not "good", then we can change it into good ones w.r.t. to J. This process can be continued until all the monomials appearing in the linear combination are good, then we get our claim. Second, any factor of a good word is a good word. Otherwise, if $u=u_1u_2\cdots u_n$ is a good word but a factor of it, say u_i, is not good, then we have that $u_i=\sum_{m\prec u_i}a_m m \pmod J$ such that

$$u=u_1\cdots u_{i-1}(\sum_{m\prec u_i}a_m m)u_{i+1}\cdots u_n \pmod J.$$

That means u is not a good word. It is a contradiction. In view of Theorem 2.3, we get the result. $\square$

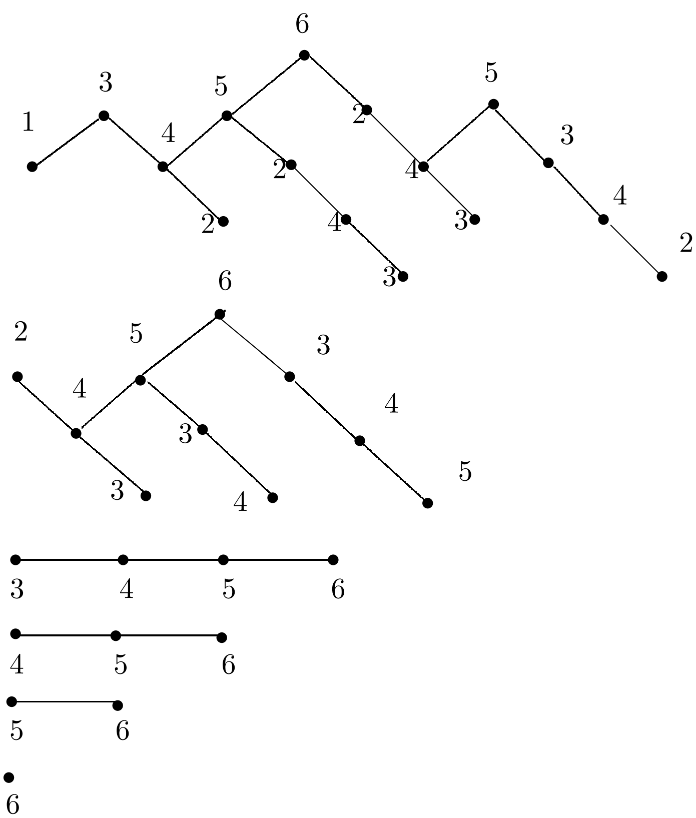
More precisely, we have the following inductive construction. For each pair of homogeneous elements $u\in U^+_\zeta, v\in U^+_\eta$, we fix the notation $p_{\zeta\eta}=\langle\omega'_\eta,\omega_\zeta\rangle$, and define a bilinear skew commutator named (r,s)-*bracketing* on the set of graded homogeneous noncommutative polynomials u, v by the formula

$$\lceil u,v\rfloor=uv-p_{\zeta\eta}vu=uv-\langle\omega'_\eta,\omega_\zeta\rangle vu.$$

We call $\lceil u \rfloor$ a *good letter* (or say, a *quantum root vector*) in U^+ if u is a good Lyndon word. By induction, we define $\lceil u \rfloor$ as

$$\lceil u \rfloor = \lceil \lceil v \rfloor \lceil w \rfloor \rfloor, \quad \text{if } \ u = vw \ \text{ is a Lyndon decomposition.}$$

We list all the good Lyndon words ordered by $<$ and the figure of them as follows



$$\begin{array}{llllllllll}
E_1 & E_{13} & E_{134} & E_{1342} & E_{1345} & E_{13452} & E_{134524} & E_{1345243} & E_{13456} & E_{134562} \\
 & E_{1345624} & E_{13456243} & E_{13456245} & E_{134562453} & E_{1345624534} & E_{13456245342} & & & \\
E_2 & E_{24} & E_{243} & E_{245} & E_{2453} & E_{24534} & E_{2456} & E_{24563} & & \\
 & E_{245634} & E_{2456345} & & & & & & & \\
E_3 & E_{34} & E_{345} & E_{3456} & & & & & & \\
E_4 & E_{45} & E_{456} & & & & & & & \\
E_5 & E_{56} & & & & & & & & \\
E_6 & & & & & & & & &
\end{array}$$

where $E_{i_1 \cdots i_n}$ denotes $e_{i_1} e_{i_2} \cdots e_{i_n}$.

Denote $\mathcal{E}_{\beta_1} = \lceil E_1 \rfloor, \mathcal{E}_{\beta_2} = \lceil E_{13} \rfloor, \mathcal{E}_{\beta_3} = \lceil E_{134} \rfloor, \cdots, \mathcal{E}_{\beta_{36}} = \lceil E_6 \rfloor$, where β_i denotes a root in Φ^+. Then we have the following theorem.

Theorem 2.6. *The set of products $\mathcal{E}_{\beta_{36}}^{n_{36}} \cdots \mathcal{E}_{\beta_2}^{n_2} \mathcal{E}_{\beta_1}^{n_1}$ is a convex PBW-type basis of $U_{r,s}(\mathfrak{n}^+)$, which is a Lyndon basis with the "convexity property" in the sense of [R3], where $n_1, \cdots, n_{36}$ are nonnegative integers.*

The proof is similar to that in [K1].

Similarly, we define a bilinear skew commutator on the set of graded homogeneous noncommutative polynomials in $U_{r,s}^-(\mathfrak{n})$. For each pair of homogeneous elements u, v in the free algebra $\mathbb{K}\langle f_1, \cdots, f_6\rangle$ and $u \in U_\zeta^-, v \in U_\eta^-$, we fix the notation ${p'}_{\zeta\eta} = \langle \omega'_\zeta, \omega_\eta\rangle^{-1}$,

$$\lceil u, v\rfloor = vu - {p'}_{\zeta\eta} uv = vu - \langle \omega'_\zeta, \omega_\eta\rangle^{-1} uv.$$

We call $\lceil u\rfloor$ a *good letter* if u is a good Lyndon word. By induction, we define $\lceil u\rfloor$ as

$$\lceil u\rfloor = \lceil \lceil v\rfloor \lceil w\rfloor \rfloor,$$

where $u = vw$ is a Lyndon decomposition. Denote $f_{i_1} f_{i_2} \cdots f_{i_n}$ by $F_{i_1 \cdots i_n}$ and set $\mathcal{F}_{\beta_1} = \lceil F_1\rfloor, \mathcal{F}_{\beta_2} = \lceil F_{13}\rfloor, \mathcal{F}_{\beta_3} = \lceil F_{134}\rfloor, \cdots, \mathcal{F}_{\beta_{36}} = \lceil F_6\rfloor$. The set of products $\mathcal{F}_{\beta_{36}}^{n_{36}} \mathcal{F}_{\beta_{35}}^{n_{35}} \cdots \mathcal{F}_{\beta_1}^{n_1}$ is a basis of $U_{r,s}(\mathfrak{n}^-)$, where $n_1, \cdots, n_{36}$ are nonnegative integers.

3. Drinfeld double and universal R-matrix

In this section, we will give the Drinfeld double structure of the algebra $U_{r,s}(\mathfrak{g})$ after preparing some of Lemmas. This structure, together with the result about the convex PBW-type basis, will be used to construct the explicit form of the canonical element and the universal R-matrix of $U_{r,s}(\mathfrak{g})$.

Lemma 3.1. $\Delta(\mathcal{E}_{\beta_i}) = \mathcal{E}_{\beta_i} \otimes 1 + \omega_{\mathcal{E}_{\beta_i}} \otimes \mathcal{E}_{\beta_i} + \sum (*) \mathcal{E}_{\beta_i}^{(1)} \omega_{\mathcal{E}_{\beta_i}^{(2)}} \otimes \mathcal{E}_{\beta_i}^{(2)}$, *where $deg(\mathcal{E}_{\beta_i}) = deg(\mathcal{E}_{\beta_i}^{(1)}) + deg(\mathcal{E}_{\beta_i}^{(2)})$, $\mathcal{E}_{\beta_i}^{(1)}$ $(< \mathcal{E}_{\beta_i})$ is a good letter, and $\mathcal{E}_{\beta_i}^{(2)}$ is a non increasing product of good letters (i.e., (r,s)-bracketing of Lyndon words), which are bigger than $\mathcal{E}_{\beta_i}$ w.r.t. the ordering $<$.*

Proof. We will prove it by induction. Assume that the Lyndon decomposition of $\mathcal{E}_{\beta_i}$ is $\mathcal{E}_{\beta_i} = \lceil \mathcal{E}_{i1}, \mathcal{E}_{i2}\rfloor$ and $\mathcal{E}_{i1}, \mathcal{E}_{i2}$ satisfying

$$\Delta(\mathcal{E}_{i1}) = \mathcal{E}_{i1} \otimes 1 + \omega_{\mathcal{E}_{i1}} \otimes \mathcal{E}_{i1} + \sum (*) \mathcal{E}_{i1}^{(1)} \omega_{\mathcal{E}_{i1}^{(2)}} \otimes \mathcal{E}_{i1}^{(2)},$$
$$\Delta(\mathcal{E}_{i2}) = \mathcal{E}_{i2} \otimes 1 + \omega_{\mathcal{E}_{i2}} \otimes \mathcal{E}_{i2} + \sum (*) \mathcal{E}_{i2}^{(1)} \omega_{\mathcal{E}_{i2}^{(2)}} \otimes \mathcal{E}_{i2}^{(2)},$$

where $\mathcal{E}_{i1}^{(2)}$'s are non increasing products of good letters $(> \mathcal{E}_{i1})$, $\mathcal{E}_{i2}^{(2)}$'s are non increasing products of good letters $(> \mathcal{E}_{i2})$. Then we have

$$\begin{aligned}
\Delta(\mathcal{E}_{\beta_i}) = {} & \big(\mathcal{E}_{i1} \otimes 1 + \omega_{\mathcal{E}_{i1}} \otimes \mathcal{E}_{i1} + \sum (*) \mathcal{E}_{i1}^{(1)} \omega_{\mathcal{E}_{i1}^{(2)}} \otimes \mathcal{E}_{i1}^{(2)}\big) \cdot \\
& \big(\mathcal{E}_{i2} \otimes 1 + \omega_{\mathcal{E}_{i2}} \otimes \mathcal{E}_{i2} + \sum (*) \mathcal{E}_{i2}^{(1)} \omega_{\mathcal{E}_{i2}^{(2)}} \otimes \mathcal{E}_{i2}^{(2)}\big) \\
& - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \big(\mathcal{E}_{i2} \otimes 1 + \omega_{\mathcal{E}_{i2}} \otimes \mathcal{E}_{i2} + \sum (*) \mathcal{E}_{i2}^{(1)} \omega_{\mathcal{E}_{i2}^{(2)}} \otimes \mathcal{E}_{i2}^{(2)}\big) \cdot \\
& \big(\mathcal{E}_{i1} \otimes 1 + \omega_{\mathcal{E}_{i1}} \otimes \mathcal{E}_{i1} + \sum (*) \mathcal{E}_{i1}^{(1)} \omega_{\mathcal{E}_{i1}^{(2)}} \otimes \mathcal{E}_{i1}^{(2)}\big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{E}_{i1}\mathcal{E}_{i2}\otimes 1 + \omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2} + \mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2}\\
&+ \sum(*)\mathcal{E}_{i1}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)} + \omega_{\mathcal{E}_{i1}}\mathcal{E}_{i2}\otimes \mathcal{E}_{i1} + \sum(*)\omega_{\mathcal{E}_{i1}}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2}^{(2)}\\
&+ \sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\mathcal{E}_{i2}\otimes \mathcal{E}_{i1}^{(2)} + \sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i1}^{(2)}\mathcal{E}_{i2}\\
&+ \sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)}\mathcal{E}_{i2}^{(2)}\\
&- \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \big(\mathcal{E}_{i2}\mathcal{E}_{i1}\otimes 1 + \omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2}\mathcal{E}_{i1} + \omega_{\mathcal{E}_{i2}}\mathcal{E}_{i1}\otimes \mathcal{E}_{i2}\\
&+ \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\mathcal{E}_{i1}\otimes \mathcal{E}_{i2}^{(2)} + \mathcal{E}_{i2}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i1} + \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1}\\
&+ \sum(*)\mathcal{E}_{i2}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)} + \sum(*)\omega_{\mathcal{E}_{i2}}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i2}\mathcal{E}_{i1}^{(2)}\\
&+ \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1}^{(2)}\big)\\
&= (\mathcal{E}_{i1}\mathcal{E}_{i2}\otimes 1 - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}\mathcal{E}_{i1}\otimes 1) + (\omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2}\mathcal{E}_{i1})\\
&+ (\mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \omega_{\mathcal{E}_{i2}}\mathcal{E}_{i1}\otimes \mathcal{E}_{i2}) + (\sum(*)\mathcal{E}_{i1}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle\\
&\cdot \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\mathcal{E}_{i1}\otimes \mathcal{E}_{i2}^{(2)}) + (\omega_{\mathcal{E}_{i1}}\mathcal{E}_{i2}\otimes \mathcal{E}_{i1} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i1})\\
&+ (\sum(*)\omega_{\mathcal{E}_{i1}}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1})\\
&+ (\sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\mathcal{E}_{i2}\otimes \mathcal{E}_{i1}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\mathcal{E}_{i2}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)})\\
&+ (\sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i1}^{(2)}\mathcal{E}_{i2} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\omega_{\mathcal{E}_{i2}}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i2}\mathcal{E}_{i1}^{(2)})\\
&+ (\sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)}\mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1}^{(2)})\\
&= (\mathcal{E}_{i1}\mathcal{E}_{i2}\otimes 1 - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}\mathcal{E}_{i1}\otimes 1) + (\omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2}\mathcal{E}_{i1})\\
&+ (\mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle\langle \omega'_{\mathcal{E}_{i1}}, \omega_{\mathcal{E}_{i2}}\rangle \mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2})\\
&+ (\sum(*)\mathcal{E}_{i1}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\langle \omega'_{\mathcal{E}_{i1}}, \omega_{\mathcal{E}_{i2}^{(2)}}\rangle \mathcal{E}_{i2}^{(1)}\mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)})\\
&+ (\langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i1} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i1})\\
&+ (\sum(*)\langle \omega'_{\mathcal{E}_{i2}^{(1)}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1})\\
&+ (\sum(*)\langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}^{(2)}}\rangle \mathcal{E}_{i1}^{(1)}\mathcal{E}_{i2}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\mathcal{E}_{i2}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)})\\
&+ (\sum(*)\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i1}^{(2)}\mathcal{E}_{i2} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\langle \omega'_{\mathcal{E}_{i1}^{(1)}}, \omega_{\mathcal{E}_{i2}}\rangle \mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i2}}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i2}\mathcal{E}_{i1}^{(2)})\\
&+ (\sum(*)\langle \omega'_{\mathcal{E}_{i2}^{(1)}}, \omega_{\mathcal{E}_{i1}^{(2)}}\rangle \mathcal{E}_{i1}^{(1)}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i1}^{(2)}}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}^{(2)}\mathcal{E}_{i2}^{(2)}\\
&- \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\langle \omega'_{\mathcal{E}_{i1}^{(1)}}, \omega_{\mathcal{E}_{i2}^{(2)}}\rangle \mathcal{E}_{i2}^{(1)}\mathcal{E}_{i1}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\omega_{\mathcal{E}_{i1}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1}^{(2)})\\
&= \mathcal{E}_{\beta_i}\otimes 1 + \omega_{\mathcal{E}_{\beta_i}}\otimes \mathcal{E}_{\beta_i} + (1-r^{-1}s)\mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}}\otimes \mathcal{E}_{i2}\\
&+ (\sum(*)\mathcal{E}_{i1}\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\langle \omega'_{\mathcal{E}_{i1}}, \omega_{\mathcal{E}_{i2}^{(2)}}\rangle \mathcal{E}_{i2}^{(1)}\mathcal{E}_{i1}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i2}^{(2)}) + 0\\
&+ (\sum(*)\langle \omega'_{\mathcal{E}_{i2}^{(1)}}, \omega_{\mathcal{E}_{i1}}\rangle \mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i1}}\omega_{\mathcal{E}_{i2}^{(2)}}\otimes \mathcal{E}_{i1}\mathcal{E}_{i2}^{(2)} - \langle \omega'_{\mathcal{E}_{i2}}, \omega_{\mathcal{E}_{i1}}\rangle \sum(*)\mathcal{E}_{i2}^{(1)}\omega_{\mathcal{E}_{i2}^{(2)}}\omega_{\mathcal{E}_{i1}}\otimes \mathcal{E}_{i2}^{(2)}\mathcal{E}_{i1})
\end{aligned}
$$

$$+ (\sum(*)\langle\omega'_{\mathcal{E}_{i2}},\omega_{\mathcal{E}^{(2)}_{i1}}\rangle\mathcal{E}^{(1)}_{i1}\mathcal{E}_{i2}\omega_{\mathcal{E}^{(2)}_{i1}}\otimes\mathcal{E}^{(2)}_{i1} - \langle\omega'_{\mathcal{E}_{i2}},\omega_{\mathcal{E}_{i1}}\rangle\sum(*)\mathcal{E}_{i2}\mathcal{E}^{(1)}_{i1}\omega_{\mathcal{E}^{(2)}_{i1}}\otimes\mathcal{E}^{(2)}_{i1})$$
$$+ (\sum(*)\mathcal{E}^{(1)}_{i1}\omega_{\mathcal{E}^{(2)}_{i1}}\omega_{\mathcal{E}_{i2}}\otimes\mathcal{E}^{(2)}_{i1}\mathcal{E}_{i2} - \langle\omega'_{\mathcal{E}_{i2}},\omega_{\mathcal{E}_{i1}}\rangle\sum(*)\langle\omega'_{\mathcal{E}^{(1)}_{i1}},\omega_{\mathcal{E}_{i2}}\rangle\mathcal{E}^{(1)}_{i1}\omega_{\mathcal{E}_{i2}}\omega_{\mathcal{E}^{(2)}_{i1}}\otimes\mathcal{E}_{i2}\mathcal{E}^{(2)}_{i1})$$
$$+ (\sum(*)\langle\omega'_{\mathcal{E}^{(1)}_{i2}},\omega_{\mathcal{E}^{(2)}_{i1}}\rangle\mathcal{E}^{(1)}_{i1}\mathcal{E}^{(1)}_{i2}\omega_{\mathcal{E}^{(2)}_{i1}}\omega_{\mathcal{E}^{(2)}_{i2}}\otimes\mathcal{E}^{(2)}_{i1}\mathcal{E}^{(2)}_{i2}$$
$$- \langle\omega'_{\mathcal{E}_{i2}},\omega_{\mathcal{E}_{i1}}\rangle\sum(*)\langle\omega'_{\mathcal{E}^{(1)}_{i1}},\omega_{\mathcal{E}^{(2)}_{i2}}\rangle\mathcal{E}^{(1)}_{i2}\mathcal{E}^{(1)}_{i1}\omega_{\mathcal{E}^{(2)}_{i2}}\omega_{\mathcal{E}^{(2)}_{i1}}\otimes\mathcal{E}^{(2)}_{i2}\mathcal{E}^{(2)}_{i1})$$
$$= \mathcal{E}_{\beta_i}\otimes 1 + \omega_{\mathcal{E}_{\beta_i}}\otimes\mathcal{E}_{\beta_i} + \sum(*)\mathcal{E}^{(1)}_{\beta_i}\omega_{\mathcal{E}^{(2)}_{\beta_i}}\otimes\mathcal{E}^{(2)}_{\beta_i}.$$

We can give some notes for the discussion above. Since any Lyndon word is smaller than its proper right factors, we can deduce that $\mathcal{E}_{i2}, \mathcal{E}^{(2)}_{i2}, \mathcal{E}^{(2)}_{i1}, \mathcal{E}^{(2)}_{i1}\mathcal{E}_{i2}, \mathcal{E}_{i2}\mathcal{E}^{(2)}_{i1}, \mathcal{E}^{(2)}_{i1}\mathcal{E}^{(2)}_{i2}$, and $\mathcal{E}^{(2)}_{i2}\mathcal{E}^{(2)}_{i1}$ can be written as non increasing products of good letters, which are bigger than $\mathcal{E}_{\beta_i}$. So we get the lemma. □

Assume $\mathcal{B}$ is the Hopf subalgebra of $U_{r,s}(\mathfrak{g})$ generated by $e_i, \omega_i^{\pm 1}$ $(1 \le i \le 6)$, and $\mathcal{B}'$ is the Hopf subalgebra of $U_{r,s}(\mathfrak{g})$ generated by $f_i, {\omega'}_i^{\pm 1}$ $(1 \le i \le 6)$.

Let us introduce linear forms η_{β_i} and γ_i in $\mathcal{B}^*$, defined by

$$\eta_{\beta i} = \sum_{g\in G(\mathcal{B})}(\mathcal{E}_{\beta i}g)^*, \qquad \gamma_i(\omega_j) = \langle\omega'_i,\omega_j\rangle, \qquad \gamma_i(e_j) = 0,$$

where $G(\mathcal{B})$ is the abelian group generated by ω_i $(1 \le i \le 6)$, and the asterisk denotes the dual basis element relative to the PBW-type basis of $\mathcal{B}$. The isomorphism $\phi : \mathcal{B}'^{coop} \to \mathcal{B}^*$ is defined by

$$\phi(\omega'_i) = \gamma_i, \qquad \phi(f_i) = \eta_i.$$

First, we will check that ϕ is a Hopf algebra homomorphism, and then we will show that it is a bijection.

Now we give a series of Lemmas, with some ideas benefited from [R1].

Lemma 3.2. $\gamma_i\eta_j\gamma_i^{-1} = \langle\omega'_i,\omega_j\rangle\eta_j$.

Proof. First, we should note that γ_i's are invertible elements in $\mathcal{B}^*$ and they are commutative with one another. It is also not difficult to see that the action of $\gamma_i\eta_j\gamma_i^{-1}$ is nonzero only on basis elements of the form $e_j\omega_1^{k_1}\cdots\omega_6^{k_6}$, and on these elements it takes the same value

$$\begin{aligned}
& \gamma_i\eta_j\gamma_i^{-1}(e_j\omega_1^{k_1}\cdots\omega_6^{k_6}) \\
= & \ \gamma_i\otimes\eta_j\otimes\gamma_i^{-1}((e_j\otimes 1\otimes 1 + \omega_j\otimes e_j\otimes 1 + \omega_j\otimes\omega_j\otimes e_j)(\omega_1^{k_1}\cdots\omega_6^{k_6})^{\otimes 3}) \\
= & \ \gamma_i(\omega_j\omega_1^{k_1}\cdots\omega_6^{k_6})\eta_j(e_j\omega_1^{k_1}\cdots\omega_6^{k_6})\gamma_i^{-1}(\omega_1^{k_1}\cdots\omega_6^{k_6}) \\
= & \ \gamma_i(\omega_j) = \langle\omega'_i,\omega_j\rangle.
\end{aligned}$$

Observing that $\eta_j(e_j\omega_1^{k_1}\cdots\omega_6^{k_6}) = 1$, we have $\gamma_i\eta_j\gamma_i^{-1} = \langle\omega'_i,\omega_j\rangle\eta_j$. □

Lemma 3.3. $\Delta(\eta_i) = \eta_i\otimes 1 + \gamma_i\otimes\eta_i$.

Proof. Since the coproduct keeps the degree, there are only two kinds of basis elements of $\mathcal{B}\otimes\mathcal{B}$ on which $\Delta(\eta_i)$ is nonzero. They are $e_i\omega_1^{j_1}\cdots\omega_6^{j_6}\otimes\omega_1^{k_1}\cdots\omega_6^{k_6}$ and $\omega_1^{j_1}\cdots\omega_6^{j_6}\otimes e_i\omega_1^{k_1}\cdots\omega_6^{k_6}$. Calculating the actions of $\Delta(\eta_i)$ on them, we get

$$\begin{aligned}\Delta(\eta_i)(e_i\omega_1^{j_1}\cdots\omega_6^{j_6}\otimes\omega_1^{k_1}\cdots\omega_6^{k_6}) &= \eta_i(e_i\omega_1^{j_1+k_1}\cdots\omega_6^{j_6+k_6})=1,\\ \Delta(\eta_i)(\omega_1^{j_1}\cdots\omega_6^{j_6}\otimes e_i\omega_1^{k_1}\cdots\omega_6^{k_6}) &= \eta_i(\omega_1^{j_1}\cdots\omega_6^{j_6}e_i\omega_1^{k_1}\cdots\omega_6^{k_6})=\langle\omega_i',\omega_1^{j_1}\cdots\omega_6^{j_6}\rangle.\end{aligned}$$

Correspondingly, we have

$$\begin{aligned}(\eta_i\otimes 1+\gamma_i\otimes\eta_i)(e_i\omega_1^{j_1}\cdots\omega_6^{j_6}\otimes\omega_1^{k_1}\cdots\omega_6^{k_6}) &= 1,\\ (\eta_i\otimes 1+\gamma_i\otimes\eta_i)(\omega_1^{j_1}\cdots\omega_6^{j_6}\otimes e_i\omega_1^{k_1}\cdots\omega_6^{k_6}) &= \langle\omega_i',\omega_1^{j_1}\cdots\omega_6^{j_6}\rangle.\end{aligned}$$

So we get $\Delta(\eta_i)=\eta_i\otimes 1+\gamma_i\otimes\eta_i$. □

Lemma 3.4.

$$\begin{aligned}&\text{(i)}\quad \eta_i\eta_j-r^{-1}\eta_j\eta_i=(1-r^{-1}s)\eta_{\alpha_i+\alpha_j},\quad \text{if }\ a_{ij}=-1,\ \text{and}\ \ i<j;\\ &\qquad\ \eta_i\eta_j=\eta_j\eta_i,\quad \text{if }\ a_{ij}=0.\\ &\text{(ii)}\quad \eta_i^2\eta_j-(r^{-1}+s^{-1})\eta_i\eta_j\eta_i+(r^{-1}s^{-1})\eta_i\eta_j^2=0,\quad \text{if }\ a_{ij}=-1,\ \text{and}\ \ i<j;\\ &\qquad\ \eta_i^2\eta_j-(r+s)\eta_i\eta_j\eta_i+(rs)\eta_i\eta_j^2=0,\quad \text{if }\ a_{ij}=-1,\ \text{and}\ \ i>j.\end{aligned}$$

Proof. We will give the proofs of the first identity in (i) and the first one in (ii), and the proofs of the others are similar. We can also ignore the ω_i's in basis since they carry no weight on η. So in our proof we can assume that $a_{ij}=-1$ and $i<j$ which implies $\alpha_i+\alpha_j\in\Phi^+$ and $e_je_i,\lceil e_ie_j\rfloor$ are in the basis. It is also clear from the definition of η that $\eta_j\eta_i(e_je_i)=1$ and zero on the other monomials, $\eta_{\alpha_i+\alpha_j}(\lceil e_ie_j\rfloor)=1$ and zero on the other monomials. In order to get the first identity, we need to compute the actions of $\eta_i\eta_j$ as follows

$$\begin{gathered}\eta_i\eta_j(e_je_i)=(\eta_i\otimes\eta_j)(\omega_je_i\otimes e_j)=\langle\omega_i',\omega_j\rangle\cdot 1=r^{-1},\\ \eta_i\eta_j(\lceil e_ie_j\rfloor)=\eta_i\eta_j(e_ie_j-se_je_i)=(1-r^{-1}s),\end{gathered}$$

where we used Lemma 3.3. Then we have $\eta_i\eta_j-r^{-1}\eta_j\eta_i=(1-r^{-1}s)\eta_{\alpha_i+\alpha_j}$. Left (resp. right) multiplied by η_i on both sides of the first identity in (i), we get

$$\begin{aligned}\eta_i^2\eta_j-r^{-1}\eta_i\eta_j\eta_i &= (1-r^{-1}s)\eta_i\eta_{\alpha_i+\alpha_j},\\ -s^{-1}\eta_i\eta_j\eta_i+r^{-1}s^{-1}\eta_j\eta_i^2 &= (1-r^{-1}s)(-s^{-1})\eta_{\alpha_i+\alpha_j}\eta_i.\end{aligned}$$

Adding the two identities together, we have

$$\eta_i^2\eta_j-(r^{-1}+s^{-1})\eta_i\eta_j\eta_i+(r^{-1}s^{-1})\eta_i\eta_j^2=(1-r^{-1}s)(\eta_i\eta_{\alpha_i+\alpha_j}-s^{-1}\eta_{\alpha_i+\alpha_j}\eta_i).$$

Since $2\alpha_i+\alpha_j$ is not a root, the only element in basis on which $\eta_{\alpha_i+\alpha_j}\eta_i$ and $\eta_i\eta_{\alpha_i+\alpha_j}$ act nontrivially is $\mathcal{E}_{\alpha_i+\alpha_j}e_i$. Observing that

$$\eta_i\eta_{\alpha_i+\alpha_j}(\mathcal{E}_{\alpha_i+\alpha_j}e_i)=s^{-1},\quad \eta_{\alpha_i+\alpha_j}\eta_i(\mathcal{E}_{\alpha_i+\alpha_j}e_i)=1,$$

we have

$$\eta_i\eta_{\alpha_i+\alpha_j} - s^{-1}\eta_{\alpha_i+\alpha_j}\eta_i = 0.$$

So we get the relation

$$\eta_i^2\eta_j - (r^{-1}+s^{-1})\eta_i\eta_j\eta_i + (r^{-1}s^{-1})\eta_i\eta_j^2 = 0. \qquad \square$$

Now we want to discuss the relations between η_{β_i} and $\mathcal{F}_{\beta_i}$, where $\beta_i \in \Phi^+$. We can identify $\mathcal{B}^*$ with $\mathcal{B}'$ with the opposite comultiplication. Let Δ' denote this opposite comultiplication and S' the antipode. With Lemmas 3.2–3.4, we have a map $\mathcal{B}^* \longrightarrow \mathcal{B}'$

$$\eta_i \mapsto (s-r)f_i \qquad \gamma_i \mapsto \omega_i'.$$

Since this map is bijective, it is an isomorphism of Hopf algebras.

Definition 3.5.([BGH1]) *For any two skew-paired Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ by a skew-dual pairing $\langle,\rangle$, one may form the Drinfel'd double $\mathcal{D}(\mathcal{A},\mathcal{U})$, which is a Hopf algebra whose underlying coalgebra is $\mathcal{A}\otimes\mathcal{U}$ with the tensor product coalgebra structure and algebra structure is defined by*

$$(a\otimes f)(a'\otimes f') = \sum \langle S_{\mathcal{U}}(f_{(1)}), a'_{(1)}\rangle\langle f_{(3)}, a'_{(3)}\rangle a a'_{(2)} \otimes f_{(2)} f',$$

for $a, a' \in \mathcal{A}$ and $f, f' \in \mathcal{U}$, and the antipode S is given by

$$S(a\otimes f) = (1\otimes S_{\mathcal{U}}(f))(S_{\mathcal{A}}(a)\otimes 1).$$

Similar to [BW1,3] and [BGH1], we have

Theorem 3.6. *The two-parameter quantum group $U = U_{r,s}(\mathfrak{g})$ is isomorphic to the Drinfel'd quantum double $\mathcal{D}(\mathcal{B},\mathcal{B}')$.*

Proof. Denote the image $e_i\otimes 1$ of e_i in $\mathcal{D}(\mathcal{B},\mathcal{B}')$ by $\check{e}_i$, and similarly for ω_i, η_i, and γ_i. Let $\varphi : \mathcal{D}(\mathcal{B},\mathcal{B}') \longrightarrow U = U_{r,s}(\mathfrak{g})$ be a map defined by:

$$\varphi(\check{e}_i) = e_i, \qquad \varphi(\check{\eta}_i) = (s-r)f_i,$$
$$\varphi(\check{\omega}_i^{\pm 1}) = \omega_i^{\pm 1}, \qquad \varphi(\check{\gamma}_i^{\pm 1}) = {\omega'}_i^{\pm 1},$$

From the above Lemmas, it is clear that φ keeps the relations in $\mathcal{B}$ and $\mathcal{B}'$. It remains to check the mixed relations (E4). Note that

$$\Delta^{(2)}(e_i) = e_i\otimes 1\otimes 1 + \omega_i\otimes e_i\otimes 1 + \omega_i\otimes\omega_i\otimes e_i,$$
$$(\Delta^{(2)op})(\eta_j) = 1\otimes 1\otimes \eta_j + 1\otimes\eta_j\otimes\gamma_j + \eta_j\otimes\gamma_j\otimes\gamma_j.$$

Using the multiplication rule in $\mathcal{D}(\mathcal{B},\mathcal{B}')$, we get

$$\check{\eta}_j\check{e}_i = \delta_{i,j}(\check{\omega}_i + \check{e}_i\check{\eta}_j - \check{\gamma}_j), \quad \textit{or} \quad [\check{e}_i, \check{\eta}_j] = \delta_{i,j}(\check{\gamma}_i - \check{\omega}_i).$$

Under φ, this corresponds to the relation

$$[e_i,(s-r)f_j]=\delta_{i,j}(\omega_i'-\omega_i),\quad or\quad [e_i,f_j]=\delta_{i,j}\frac{\omega_i-\omega_i'}{r-s},$$

which is $(E4)$. $\square$

Lemma 3.7. *Let β_i denote a root in Φ^+ w.r.t. the ordering $<$, and $\mathcal{F}_{\beta_i}=\lceil\lceil\mathcal{F}_{\beta_{i1}}\rfloor\lceil\mathcal{F}_{\beta_{i2}}\rfloor\rfloor$, then $\eta_{\beta_i}=c_{\beta_i}\mathcal{F}_{\beta_i}$, where c_{β_i} satisfies*

$$c_{\beta_i}=-\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle(1-\langle{\omega'}_{\beta_{i2}},\omega_{\beta_{i1}}\rangle\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle)^{-1}c_{\beta_{i1}}c_{\beta_{i2}}.$$

Proof. Assume that $\eta_{\beta_i}=a\eta_{\beta_{i1}}\eta_{\beta_{i2}}+b\eta_{\beta_{i2}}\eta_{\beta_{i1}}$. Calculating the actions of both sides of the equation on elements $\mathcal{E}_{\beta_i}$ and $\mathcal{E}_{\beta_{i2}}\mathcal{E}_{\beta_{i1}}$, we have

$$\begin{aligned}
\eta_{\beta_i}(\mathcal{E}_{\beta_i}) &= 1,\\
\eta_{\beta_{i1}}\eta_{\beta_{i2}}(\mathcal{E}_{\beta_i}) &= 1-\langle{\omega'}_{\beta_{i2}},\omega_{\beta_{i1}}\rangle\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle,\\
\eta_{\beta_{i2}}\eta_{\beta_{i1}}(\mathcal{E}_{\beta_i}) &= 0,\\
\eta_{\beta_i}(\mathcal{E}_{\beta_{i2}}\mathcal{E}_{\beta_{i1}}) &= 0,\\
\eta_{\beta_{i1}}\eta_{\beta_{i2}}(\mathcal{E}_{\beta_{i2}}\mathcal{E}_{\beta_{i1}}) &= \langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle,\\
\eta_{\beta_{i2}}\eta_{\beta_{i1}}(\mathcal{E}_{\beta_{i2}}\mathcal{E}_{\beta_{i1}}) &= 1.
\end{aligned}$$

Then we have

$$\begin{aligned}
\eta_{\beta_i} &= (1-\langle{\omega'}_{\beta_{i2}},\omega_{\beta_{i1}}\rangle\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle)^{-1}\eta_{\beta_{i1}}\eta_{\beta_{i2}}\\
&\quad -\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle(1-\langle{\omega'}_{\beta_{i2}},\omega_{\beta_{i1}}\rangle\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle)^{-1}\eta_{\beta_{i2}}\eta_{\beta_{i1}}\\
&= -\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle(1-\langle{\omega'}_{\beta_{i2}},\omega_{\beta_{i1}}\rangle\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle)^{-1}\\
&\quad (c_{\beta_{i1}}c_{\beta_{i2}}\mathcal{F}_{\beta_{i2}}\mathcal{F}_{\beta_{i1}}-\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle^{-1}c_{\beta_{i1}}c_{\beta_{i2}}\mathcal{F}_{\beta_{i1}}\mathcal{F}_{\beta_{i2}})\\
&= -\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle(1-\langle{\omega'}_{\beta_{i2}},\omega_{\beta_{i1}}\rangle\langle{\omega'}_{\beta_{i1}},\omega_{\beta_{i2}}\rangle)^{-1}c_{\beta_{i1}}c_{\beta_{i2}}\mathcal{F}_{\beta_i}\\
&= c_{\beta_i}\mathcal{F}_{\beta_i}. \qquad \square
\end{aligned}$$

Lemma 3.8.

(i) $$\langle\eta^n_{\beta i},\mathcal{E}^{n'}_{\beta i}\rangle=\delta_{n,n'}\frac{\Psi_n(rs^{-1})}{(1-rs^{-1})^n},$$

(ii) $$\langle\eta^{n_{36}}_{\beta 36}\cdots\eta^{n_2}_{\beta 2}\eta^{n_1}_{\beta 1},\,\mathcal{E}^{n'_{36}}_{\beta 36}\cdots\mathcal{E}^{n'_2}_{\beta 2}\mathcal{E}^{n'_1}_{\beta 1}\rangle=\prod_{i=1}^{36}\delta_{n_i,n'_i}\frac{\Psi_{n_i}(rs^{-1})}{(1-rs^{-1})^{n_i}},$$

where $\Psi_n(a)=(1-a)(1-a^2)\cdots(1-a^n)$.

Proof. (i) By Lemma 3.1, we have

$$\Delta(\mathcal{E}_{\beta_i})=\mathcal{E}_{\beta_i}\otimes 1+\omega_{\mathcal{E}_{\beta_i}}\otimes\mathcal{E}_{\beta_i}+\sum(*)\mathcal{E}^{(1)}_{\beta_i}\omega_{\mathcal{E}^{(2)}_{\beta_i}}\otimes\mathcal{E}^{(2)}_{\beta_i},$$

where $deg(\mathcal{E}_{\beta_i})=deg(\mathcal{E}^{(1)}_{\beta_i})+deg(\mathcal{E}^{(2)}_{\beta_i})$, and $\mathcal{E}^{(2)}_{\beta_i}$ are the products of good letters, which are bigger than $\mathcal{E}_{\beta_i}$. From $\Delta(\mathcal{E}^{n'}_{\beta_i})=\sum\mathcal{E}\omega_{\overline{\mathcal{E}}}\otimes\overline{\mathcal{E}}$, we know that all the $\overline{\mathcal{E}}$'s are bigger

than $\mathcal{E}_{\beta_i}$ except for $\mathcal{E}_{\beta_i}$ itself. So the terms paired with $\eta_{\beta_i}^{n-1}\otimes\eta_{\beta_i}$ being nonzero are $\sum\mathcal{E}_{\beta_i}\cdots\mathcal{E}_{\beta_i}\omega_{\mathcal{E}_{\beta_i}}\mathcal{E}_{\beta_i}\cdots\mathcal{E}_{\beta_i}\otimes\mathcal{E}_{\beta_i}$, which gives the following

$$\begin{aligned}
\langle\eta_{\beta_i}^{n},\mathcal{E}_{\beta_i}^{n'}\rangle &= \langle\eta_{\beta_i}^{n-1}\otimes\eta_{\beta_i},\Delta(\mathcal{E}_{\beta_i}^{n'})\rangle\\
&= \langle\eta_{\beta_i}^{n-1}\otimes\eta_{\beta_i},\sum\mathcal{E}_{\beta_i}\cdots\mathcal{E}_{\beta_i}\omega_{\mathcal{E}_{\beta_i}}\mathcal{E}_{\beta_i}\cdots\mathcal{E}_{\beta_i}\otimes\mathcal{E}_{\beta_i}\rangle\\
&= (1+\langle\omega'_{\mathcal{E}_{\beta_i}},\omega_{\mathcal{E}_{\beta_i}}\rangle+\langle\omega'_{\mathcal{E}_{\beta_i}},\omega_{\mathcal{E}_{\beta_i}}\rangle^2+\cdots+\langle\omega'_{\mathcal{E}_{\beta_i}},\omega_{\mathcal{E}_{\beta_i}}\rangle^{(n'-1)})\langle\eta_{\beta_i}^{n-1},\mathcal{E}_{\beta_i}^{n'-1}\rangle\\
&= \frac{1-\langle\omega'_{\mathcal{E}_{\beta_i}},\omega_{\mathcal{E}_{\beta_i}}\rangle^{n}}{1-\langle\omega'_{\mathcal{E}_{\beta_i}},\omega_{\mathcal{E}_{\beta_i}}\rangle}\langle\eta_{\beta_i}^{n-1},\mathcal{E}_{\beta_i}^{n'-1}\rangle\\
&= \delta_{n,n'}\frac{\Psi_n(rs^{-1})}{(1-rs^{-1})^{n}}.
\end{aligned}$$

(ii) Similarly, we can use Lemma 3.1 to prove (ii). Since $\mathcal{E}_{\beta_1}<\mathcal{E}_{\beta_2}<\cdots<\mathcal{E}_{\beta_{36}}$, the terms paired with $\eta_{\beta_{36}}^{n_{36}}\eta_{\beta_{35}}^{n_{35}}\cdots\eta_{\beta_2}^{n_2}\otimes\eta_{\beta_1}^{n_1}$ being nonzero are of the form $?\otimes\mathcal{E}_{\beta_1}^{n_1}$. So we have

$$\begin{aligned}
&\langle\eta_{\beta_{36}}^{n_{36}}\cdots\eta_{\beta_2}^{n_2}\eta_{\beta_1}^{n_1},\,\mathcal{E}_{\beta_{36}}^{n'_{36}}\cdots\mathcal{E}_{\beta_2}^{n'_2}\mathcal{E}_{\beta_1}^{n'_1}\rangle\\
&\quad= \langle\eta_{\beta_{36}}^{n_{36}}\cdots\eta_{\beta_2}^{n_2}\otimes\eta_{\beta_1}^{n_1},\,\Delta(\mathcal{E}_{\beta_{36}}^{n'_{36}}\cdots\mathcal{E}_{\beta_2}^{n'_2})\Delta(\mathcal{E}_{\beta_1}^{n'_1})\rangle\\
&\quad= \frac{\Phi_{n_1}(rs^{-1})}{(1-rs^{-1})^{n_1}}\langle\eta_{\beta_{36}}^{n_{36}}\cdots\eta_{\beta_2}^{n_2},\,\mathcal{E}_{\beta_{36}}^{n'_{36}}\cdots\mathcal{E}_{\beta_2}^{n'_2}\rangle\\
&\quad= \prod_{i=1}^{36}\delta_{n_i,n'_i}\frac{\Psi_{n_i}(rs^{-1})}{(1-rs^{-1})^{n_i}}.
\end{aligned}$$

We complete the proofs. $\square$

Theorem 3.9. *The canonical element $\Theta\in U_{r,s}(\mathfrak{n}^-)\otimes U_{r,s}(\mathfrak{n}^+)$ is given by*

$$\begin{aligned}
\Theta &= \sum\frac{(1-rs^{-1})^{n_1}(1-rs^{-1})^{n_2}\cdots(1-rs^{-1})^{n_{36}}}{\Psi_{n_1}(rs^{-1})\Psi_{n_2}(rs^{-1})\cdots\Psi_{n_{36}}(rs^{-1})}\eta_{\beta_{36}}^{n_{36}}\cdots\eta_{\beta_2}^{n_2}\eta_{\beta_1}^{n_1}\otimes\mathcal{E}_{\beta_{36}}^{n_{36}}\cdots\mathcal{E}_{\beta_2}^{n_2}\mathcal{E}_{\beta_1}^{n_1}\\
&= \sum\frac{(1-rs^{-1})^{n_1+\cdots+n_{36}}c_{\beta_1}^{n_1}\cdots c_{\beta_{36}}^{n_{36}}}{\Psi_{n_1}(rs^{-1})\Psi_{n_2}(rs^{-1})\cdots\Psi_{n_{36}}(rs^{-1})}\mathcal{F}_{\beta_{36}}^{n_{36}}\cdots\mathcal{F}_{\beta_2}^{n_2}\mathcal{F}_{\beta_1}^{n_1}\otimes\mathcal{E}_{\beta_{36}}^{n_{36}}\cdots\mathcal{E}_{\beta_2}^{n_2}\mathcal{E}_{\beta_1}^{n_1}.
\end{aligned}$$

We want to describe two linear transformations $P,\widetilde{f}$, which build up the universal R-matrix $\mathcal{R}$.

(i) $P: M'\otimes M\longrightarrow M\otimes M'$ is the flip operator given by $P(m'\otimes m)=(m\otimes m')$.
(ii) $\widetilde{f}: M\otimes M'\longrightarrow M\otimes M'$ is a linear transformation based on the f defined below.

We define $f:\Lambda\times\Lambda\longrightarrow\mathbb{K}$ as

$$f(\lambda,\mu)=\langle\omega'_\mu,\omega_\lambda\rangle^{-1},$$

which satisfies

$$f(\lambda+\mu,\nu)=f(\lambda,\nu)f(\mu,\nu), \qquad f(\lambda,\mu+\nu)=f(\lambda,\mu)f(\lambda,\nu),$$
$$f(\alpha_i,\mu)=\langle\omega'_\mu,\omega_i\rangle^{-1}, \qquad f(\lambda,\alpha_i)=\langle\omega'_i,\omega_\lambda\rangle^{-1}.$$

Now we define linear transformations $\widetilde{f}=\widetilde{f}_{M,M'}: M\otimes M'\longrightarrow M\otimes M'$ by

$$\widetilde{f}(m\otimes m')=f(\lambda,\mu)(m\otimes m')$$

for $m\in M_\lambda$ and $m'\in M'_\mu$.

Proposition 3.10. *Let M and M' be any $U_{r,s}(\mathfrak{g})$-modules in category $\mathcal{O}$ (see [BW2], [BGH2]), then the map*

$$\mathcal{R}_{M',M}=\Theta\circ\widetilde{f}\circ P: M'\otimes M\longrightarrow M\otimes M'$$

is an isomorphism of $U_{r,s}(\mathfrak{g})$-modules.

The proof is similar to that of Theorem 3.4 in [BGH2]. On the other hand, it is not difficult to check that each map $\mathcal{R}_{M,M}$ satisfies the quantum Yang-Baxter equation and the braid relation with a twist.

4. Weight modules of finite-dimension

Let Λ be the weight lattice of $\mathfrak{g}$. Associated to any $\lambda\in\Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra U^0 generated by the elements ω_i,ω'_i $(1\le i\le 6)$ to $\mathbb{K}$ satisfying

$$\hat{\lambda}(\omega_i)=\langle\omega'_\lambda,\omega_i\rangle, \qquad \hat{\lambda}(\omega'_i)=\langle\omega'_i,\omega_\lambda\rangle^{-1}.$$

Let M be a module for $U_{r,s}(\mathfrak{g})$ of dimension $d<\infty$. As $\mathbb{K}$ is algebraically closed, by linear algebra, we have

$$M=\bigoplus_\chi M_\chi,$$

where each $\chi: U^0\to\mathbb{K}$ is an algebra homomorphism, and M_χ is the corresponding weight space. We say that U^0 acts semisimply on M if M can be decomposed into genuine eigenspaces relative to U^0.

We can deduce from the relations (E2) & (E3) that

$$e_jM_\chi\subseteq M_{\chi\cdot\widehat{\alpha_j}}, \quad f_jM_\chi\subseteq M_{\chi\cdot\widehat{-\alpha_j}}. \tag{4.1}$$

Lemma 4.1. *Assume that rs^{-1} is not a root of unity, and suppose $\hat{\zeta}=\hat{\eta}$ for $\zeta,\eta\in Q$, then $\zeta=\eta$.*

Proof. Assume $\zeta=\sum_{i=1}^6\zeta_i\alpha_i\in\Lambda$. By definition, we have

$$\hat{\zeta}(\omega_j)=\langle\omega'_\zeta,\omega_j\rangle=r^{\sum_{i=1}^6\zeta_ip_{ji}}s^{-\sum_{i=1}^6\zeta_iq_{ji}},$$
$$\hat{\zeta}(\omega'_j)=\langle\omega'_j,\omega_\zeta\rangle^{-1}=r^{-\sum_{i=1}^6\zeta_ip_{ij}}s^{\sum_{i=1}^6\zeta_iq_{ij}}.$$

If $\eta = \sum_{i=1}^{6} \eta_i \alpha_i$, then the condition $\hat{\zeta} = \hat{\eta}$ gives the equations

$$r^{\sum_{i=1}^{6} \zeta_i p_{ji}} s^{-\sum_{i=1}^{6} \zeta_i q_{ji}} = r^{\sum_{i=1}^{6} \eta_i p_{ji}} s^{-\sum_{i=1}^{6} \eta_i q_{ji}},$$
$$r^{-\sum_{i=1}^{6} \zeta_i p_{ij}} s^{\sum_{i=1}^{6} \zeta_i q_{ij}} = r^{-\sum_{i=1}^{6} \eta_i p_{ij}} s^{\sum_{i=1}^{6} \eta_i q_{ij}}.$$

It is not difficult to get the equations as follows

$$r^{\sum_{i=1}^{6}(\zeta_i-\eta_i)p_{ji}} s^{-\sum_{i=1}^{6}(\zeta_i-\eta_i)q_{ji}} = 1, \tag{4.2}$$

$$r^{\sum_{i=1}^{6}(\zeta_i-\eta_i)p_{ij}} s^{-\sum_{i=1}^{6}(\zeta_i-\eta_i)q_{ij}} = 1. \tag{4.3}$$

Multiplying (4.2) with (4.3), we get

$$r^{\sum_{i=1}^{6}(\zeta_i-\eta_i)(p_{ji}+p_{ij})} s^{-\sum_{i=1}^{6}(\zeta_i-\eta_i)(q_{ji}+q_{ij})} = 1. \tag{4.4}$$

By the definitions of p_{ij} and q_{ij}, and Lemma 1.1, we have

$$(rs^{-1})^{\sum_{i=1}^{6}(\zeta_i-\eta_i)(\alpha_i,\alpha_j)} = 1.$$

Due to rs^{-1} being not a root of unity, we get

$$\sum_{i=1}^{6}(\zeta_i - \eta_i)(\alpha_i, \alpha_j) = 0. \tag{4.5}$$

Since j in (4.5) is arbitrary, we get a system of homogeneous linear equations in variables $\zeta_i - \eta_i$, whose coefficient-matrix is exactly the Cartan matrix A which is invertible. Thus, we see that the system of homogeneous linear equations has only zero solution, that is, for any i, we have

$$\zeta_i - \eta_i = 0.$$

So we get the result. $\square$

Remark 4.2. Owing to Lemma 4.1, we can simplify the notation by writing M_λ (for $\lambda \in \Lambda$) as usual for the weight space instead of $M_{\hat{\lambda}}$. So it makes sense to let (4.1) take the classical forms as $e_j M_\lambda \subseteq M_{\lambda+\alpha}$ and $f_j M_\lambda \subseteq M_{\lambda-\alpha}$.

Proposition 4.3. *If M is a finite-dimensional $U_{r,s}(\mathfrak{g})$-module and rs^{-1} is not a root of unity, then the elements e_i, f_i act nilpotently on M, where $1 \le i \le 6$.*

Proof. Since M is a direct sum of its weight spaces, we only need to consider the actions of e_i, f_i on each M_λ. We know that $e_j^k.M_\lambda \subseteq M_{\lambda-k\alpha_j}$. Since $k\alpha_j$'s are distinct and M is a finite-dimensional $U_{r,s}(\mathfrak{g})$-module, e_i's act nilpotently on M_λ. So do the actions of f_i's on M. $\square$

It is not difficult to see that any simple $U_{r,s}(\mathfrak{g})$-module is a highest weight module by Proposition 4.3 and (4.1). Having Lemma 4.1 for the type E-series, one has a similar weight representation theory as in [BW1] for type A, and [BGH2] for types B, C, D.

5. Isomorphisms among quantum groups

In what follows, we will discuss the isomorphic relationship between the two-parameter quantum group and the one-parameter quantum double for type E_6. In fact, the following result with an analogous argument still holds for those of types A (with rank ≥ 3), D, and E_7, E_8.

Proposition 5.1. *Assume that there is an isomorphism of Hopf algebras $\varphi : U_{r,s}(\mathfrak{g}) \longrightarrow U_{q,q^{-1}}(\mathfrak{g})$ for some q, then $r = q$ and $s = q^{-1}$.*

Proof. Let π be the canonical surjection from $U_{q,q^{-1}}(\mathfrak{g})$ onto the standard one-parameter quantum group $U_q(\mathfrak{g})$ of [Ja] given by $\pi(e_i) = E_i$, $\pi(f_i) = F_i$, $\pi(\omega_i^{\pm 1}) = K_i^{\pm 1}$, $\pi(\omega_i'^{\pm 1}) = K_i^{\mp 1}$. For $1 \leq i \leq 6$, we have

$$\Delta(\pi\varphi(e_i)) = (\pi\varphi \otimes \pi\varphi) \circ (\Delta(e_i)). \tag{5.1}$$

Note that $\pi\varphi(e_i)$ is a skew-primitive element and $\pi\varphi(\omega_i)$ is a group-like element in $U_q(\mathfrak{g})$. The elements in the group G generated by K_i and the skew-primitive elements span the subspace

$$\sum_{j=1}^{6} (\mathbb{K}E_j + \mathbb{K}F_j) + \mathbb{K}G.$$

So we can assume that

$$\pi\varphi(e_i) = \sum_{j=1}^{6} a_{ij}E_j + b_{ij}F_j + \sum_{g \in G} c_{ig}g,$$

where $1 \leq i \leq 6$, and $a_{ij}, b_{ij}, c_{ig} \in \mathbb{K}$. Then we have

$$\Delta(\pi\varphi(e_i)) = \sum_{j=1}^{6} a_{ij}(E_j \otimes 1 + K_j \otimes E_j) + b_{ij}(1 \otimes F_j + F_j \otimes K_j^{-1}) + \sum_{g \in G} c_{ig}g \otimes g. \tag{5.2}$$

On the other hand, we have

$$(\pi\varphi \otimes \pi\varphi)(\Delta(e_i)) = \sum_{j=1}^{6} (a_{ij}E_j \otimes 1 + b_{ij}F_j \otimes 1 + \pi\varphi(\omega_i) \otimes a_{ij}E_j + \pi\varphi(\omega_i) \otimes b_{ij}F_j) \tag{5.3}$$

$$+ \sum_{g} (c_{ig}g \otimes 1 + \pi\varphi(\omega_i) \otimes c_{ig}g).$$

Observing the formula (5.1) and comparing the coefficients of the terms $? \otimes 1$ in both equations above, we have

$$\sum_{j=1}^{6} (a_{ij}E_j + c_{i1}1) = \sum_{j=1}^{6} (a_{ij}E_j + b_{ij}F_j) + \sum_{g \in G} c_{ig}g + c_{i1}\pi\varphi(\omega_i).$$

Then all $b_{ij} = 0$ and $c_{ig} = 0$ for all g except for $g \in \{1, \pi\varphi(\omega_i)\}$, and in which case we have $c_{i,\pi\varphi(\omega_i)} = -c_{i1}$. So we have

$$\pi\varphi(e_i) = \sum_{j=1}^{6} a_{ij}E_j + c_{i1}(1 - \pi\varphi(\omega_i)).$$

Thus we can simplify the right-hand sides of the equations (5.2), (5.3) and get

$$\begin{aligned}\sum_{j=1}^{6} &a_{ij}(E_j \otimes 1 + K_j \otimes E_j) + c_{i1}(1 \otimes 1 - \pi\varphi(\omega_i) \otimes \pi\varphi(\omega_i)) \\ &= \sum_{j=1}^{6} a_{ij}(E_j \otimes 1 + \pi\varphi(\omega_i) \otimes E_j) + c_{i1}(1 \otimes 1 - \pi\varphi(\omega_i) \otimes 1) + c_{i1}(\pi\varphi(\omega_i) \otimes 1 \\ &\quad - \pi\varphi(\omega_i) \otimes \pi\varphi(\omega_i)).\end{aligned}$$

This implies

$$a_{ij}(K_j - \pi\varphi(\omega_i)) = 0, \quad \text{for all } 1 \le i \le 6.$$

So all a_{ij} equal zero except for one index j. That means, the index j is related to the index i via φ. We thus let j_i indicate such a j, such that $\pi\varphi(\omega_i) = K_{j_i}$.

As $\omega_i e_k \omega_i^{-1} = r^{p_{ik}}(s^{-1})^{q_{ik}} e_k$ and by the results above, we get that

$$\begin{aligned}\pi\varphi(\omega_i e_k) &= \pi\varphi(r^{p_{ik}} s^{-q_{ik}} e_k \omega_i), \\ K_{j_i}(a_{kj_k}E_{j_k} + c_{k1}(1 - K_{j_k})) &= r^{p_{ik}} s^{-q_{ik}}(a_{kj_k}E_{j_k} + c_{k1}(1 - K_{j_k}))K_{j_i}, \\ q^{\langle \alpha_{j_i}, \alpha_{j_k}\rangle} a_{kj_k}E_{j_k}K_{j_i} + c_{k1}(1 - K_{j_k})K_{j_k} &= r^{p_{ik}} s^{-q_{ik}}(a_{kj_k}E_{j_k} + c_{k1}(1 - K_{j_k}))K_{j_i}.\end{aligned}$$

The last identity implies that $c_{k1} = 0$ and $q^{\langle \alpha_{j_i}, \alpha_{j_k}\rangle} = r^{p_{ik}} s^{-q_{ik}}$.

Since $(\alpha_{j_i}, \alpha_{j_k}) = (\alpha_i, \alpha_k)$, it is not difficult to get that $r = q$ and $s = q^{-1}$ by analyzing the three cases

$$\begin{cases} q^{\langle \alpha_{j_i}, \alpha_{j_k}\rangle} = q^2, \ r^{p_{ik}} s^{-q_{ik}} = rs^{-1}, & \text{if} \quad i = k, \\ q^{\langle \alpha_{j_i}, \alpha_{j_k}\rangle} = q^{-1}, \ r^{p_{ik}} s^{-q_{ik}} = s, & \text{if} \quad a_{ik} = -1, \ i < k, \\ q^{\langle \alpha_{j_i}, \alpha_{j_k}\rangle} = q^{-1}, \ r^{p_{ik}} s^{-q_{ik}} = r^{-1}, & \text{if} \quad a_{ik} = -1, \ i > k. \end{cases}$$

So we complete the proof. □

6. Appendix

As an interpretation of Lemma 3.1, we give an example in the following, where the good Lyndon words arising from the type E_6 case.

By virtue of the coproduct formula for the type A case, we have

$$\Delta(\mathcal{E}_{245}) = \mathcal{E}_{245} \otimes 1 + (1 - r^{-1}s)e_2\omega_{45} \otimes \mathcal{E}_{45} + (1 - r^{-1}s)\mathcal{E}_{24}\omega_5 \otimes e_5 + \omega_{245} \otimes \mathcal{E}_{245}.$$

Since $\mathcal{E}_{2453} = \lceil \mathcal{E}_{245}, e_3 \rfloor$, we have

$$
\begin{aligned}
&\Delta(\mathcal{E}_{2453}) = \Delta(\lceil \mathcal{E}_{245}, e_3 \rfloor) \\
&= (\mathcal{E}_{245} \otimes 1 + (1-r^{-1}s)e_2\omega_{45} \otimes \mathcal{E}_{45} + (1-r^{-1}s)\mathcal{E}_{24}\omega_5 \otimes e_5 + \omega_{245} \otimes \mathcal{E}_{245})\cdot \\
&\quad\ (e_3 \otimes 1 + \omega_3 \otimes e_3) - r^{-1}(e_3 \otimes 1 + \omega_3 \otimes e_3)(\mathcal{E}_{245} \otimes 1 + (1-r^{-1}s)e_2\omega_{45} \otimes \mathcal{E}_{45} \\
&\quad + (1-r^{-1}s)\mathcal{E}_{24}\omega_5 \otimes e_5 + \omega_{245} \otimes \mathcal{E}_{245}) \\
&= \mathcal{E}_{245}e_3 \otimes 1 + \mathcal{E}_{245}\omega_3 \otimes e_3 + (1-r^{-1}s)e_2\omega_{45}e_3 \otimes \mathcal{E}_{45} + (1-r^{-1}s)e_2\omega_{345} \otimes \mathcal{E}_{45}e_3 \\
&\quad + (1-r^{-1}s)\mathcal{E}_{24}\omega_5e_3 \otimes e_5 + (1-r^{-1}s)\mathcal{E}_{24}\omega_{35} \otimes e_5e_3 + \omega_{245}e_3 \otimes \mathcal{E}_{245} \\
&\quad + \omega_{2453} \otimes \mathcal{E}_{245}e_3 - r^{-1}(e_3\mathcal{E}_{245} \otimes 1 + \omega_3\mathcal{E}_{245} \otimes e_3 + (1 - r^{-1}s)e_3e_2\omega_{45} \otimes \mathcal{E}_{45} \\
&\quad + (1 - r^{-1}s)\omega_3e_2\omega_{45} \otimes e_3\mathcal{E}_{45} + (1 - r^{-1}s)e_3\mathcal{E}_{24}\omega_5 \otimes e_5 \\
&\quad + (1 - r^{-1}s)\omega_3\mathcal{E}_{24}\omega_5 \otimes e_3e_5 + e_3\omega_{245} \otimes \mathcal{E}_{245} + \omega_{2453} \otimes e_3\mathcal{E}_{245}) \\
&= \mathcal{E}_{245}e_3 \otimes 1 + \mathcal{E}_{245}\omega_3 \otimes e_3 + (1-r^{-1}s)r^{-1}e_2e_3\omega_{45} \otimes \mathcal{E}_{45} + (1-r^{-1}s)e_2\omega_{345} \otimes \mathcal{E}_{45}e_3 \\
&\quad + (1-r^{-1}s)\mathcal{E}_{24}e_3\omega_5 \otimes e_5 + (1-r^{-1}s)\mathcal{E}_{24}\omega_{35} \otimes e_5e_3 + r^{-1}e_3\omega_{245} \otimes \mathcal{E}_{245} \\
&\quad + \omega_{2453} \otimes \mathcal{E}_{245}e_3 - r^{-1}(e_3\mathcal{E}_{245} \otimes 1 + s\mathcal{E}_{245}\omega_3 \otimes e_3 + (1 - r^{-1}s)e_3e_2\omega_{45} \otimes \mathcal{E}_{45} \\
&\quad + (1 - r^{-1}s)e_2\omega_3\omega_{45} \otimes e_3\mathcal{E}_{45} + (1 - r^{-1}s)e_3\mathcal{E}_{24}\omega_5 \otimes e_5 \\
&\quad + (1 - r^{-1}s)s\mathcal{E}_{24}\omega_{35} \otimes e_3e_5 + e_3\omega_{245} \otimes \mathcal{E}_{245} + \omega_{2453} \otimes e_3\mathcal{E}_{245}) \\
&= (\mathcal{E}_{245}e_3 \otimes 1 - r^{-1}e_3\mathcal{E}_{245} \otimes 1) + (\mathcal{E}_{245}\omega_3 \otimes e_3 - r^{-1}s\mathcal{E}_{245}\omega_3 \otimes e_3) \\
&\quad + (1 - r^{-1}s)(r^{-1}e_2e_3\omega_{45} \otimes \mathcal{E}_{45} - r^{-1}e_3e_2\omega_{45} \otimes \mathcal{E}_{45}) \\
&\quad + (1 - r^{-1}s)(e_2\omega_{345} \otimes \mathcal{E}_{45}e_3 - r^{-1}e_2\omega_3\omega_{45} \otimes e_3\mathcal{E}_{45}) \\
&\quad + (1 - r^{-1}s)(\mathcal{E}_{24}e_3\omega_5 \otimes e_5 - r^{-1}e_3\mathcal{E}_{24}\omega_5 \otimes e_5) \\
&\quad + (1 - r^{-1}s)(\mathcal{E}_{24}\omega_{35} \otimes e_5e_3 - r^{-1}s\mathcal{E}_{24}\omega_{35} \otimes e_3e_5) \\
&\quad + (r^{-1}e_3\omega_{245} \otimes \mathcal{E}_{245} - r^{-1}e_3\omega_{245} \otimes \mathcal{E}_{245}) + (\omega_{2453} \otimes \mathcal{E}_{245}e_3 - r^{-1}\omega_{2453} \otimes e_3\mathcal{E}_{245}) \\
&= \mathcal{E}_{2453} \otimes 1 + (1 - r^{-1}s)\mathcal{E}_{245}\omega_3 \otimes e_3 + (1 - r^{-1}s)\mathcal{E}_{243}\omega_5 \otimes e_5 \\
&\quad + (1 - r^{-1}s)^2\mathcal{E}_{24}\omega_{35} \otimes e_5e_3 + (1 - r^{-1}s)^2e_2\omega_{345} \otimes \mathcal{E}_{45}e_3 \\
&\quad - (1 - r^{-1}s)r^{-1}e_2\omega_{345} \otimes \mathcal{E}_{345} + \omega_{2453} \otimes \mathcal{E}_{2453}.
\end{aligned}
$$

Remark. As indicated in Lemma 3.1, the right hand-side of the formula above does show that each product's ordering in the summation consisted of those possible good letters (appearing as the 2nd factors in those tensor monomials) satisfies the required non increasing property with respect to the ordering $<$.

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References

[B] G. Benkart, *Down-up algebras and Witten's deformations of the universal enveloping algebra of* $\mathfrak{sl}_2$*; Recent Progress in Algebra,* Contemp. Math., vol. 224, Amer. Math. Soc., 1998, pp. 29–45.

[BW1] G. Benkart, S. Witherspoon, *Two-parameter quantum groups and Drinfel'd doubles*. Algebra Represent. Theory 7, no. 3 (2004), 261–286.

[BW2] G. Benkart, S. Witherspoon, *Representations of two-parameter quantum groups and Schur-Weyl duality,* Hopf algebras, 65–92, Lecture Notes in Pure and Appl. Math., 237, Dekker, New York, 2004.

[BW3] G. Benkart, S. Witherspoon, *Restricted two-parameter quantum groups,* Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry, 293–318, Fields Inst. Commun., vol. 40, Amer. Math. Soc., Providence, RI, 2004.

[BGH1] N. Bergeron, Y. Gao, N. Hu, *Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups,* math.RT/0505614, J. of Algebra 301 (2006), 378–405.

[BGH2] N. Bergeron, Y. Gao, N. Hu, *Representations of two-parameter quantum orthogonal and symplectic groups,* math.QA/0510124, Contemp. Math. USA (to appear).

[HS] N. Hu, Q. Shi, *Two-parameter quantum group of exceptional type* G_2 *and Lusztig's symmetries,* math.QA/0601444, Pacific J. Math. (to appear).

[Ja] J.C. Jantzen, *Lectures on Quantum Groups*, vol. 6, Graduate Studies in Math. Amer. Math. Soc. Providence, 1996

[K1] V.K. Kharchenko, *A quantum analog of the Poincaré-Birkhoff-Witt theorem,* Algebra and Logic 38, no. 4 (1999), 259–276.

[K2] V.K. Kharchenko, *A combinatorial approach to the quantification of Lie algebras,* Pacific J. Math. 23, no. 1 (2002), 191–233.

[LR] M. Lalonde, A. Ram, *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Ams. Math. Soc., 347, no. 5 (1995), 1821–1830.

[LO] M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, **17**, Addison-Wesley Publ. Co., 1983.

[R1] M. Rosso, *An analogue of P.B.W. theorem and the universal R-matrix for* $U_h sl(N+1)$, Comm. Math. Phys. 124 (1989), 307–318.

[R2] M. Rosso, *Lyndon words and universal R-matrices*, Lecture at M.S.R.I. (1999).

[R3] M. Rosso, *Lyndon bases and the multiplicative formula for R-matrices*, (2002), preprint.

[T] M. Takeuchi, *A two-parameter quantization of $GL(n)$*, Proc. Japan. Acad. 66 Ser A (1990), 112–114.